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On the equation $x^k = z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$ in a free semigroup

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Abstract

Word equations of the form $x^k = z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$ are considered in this paper. In particular, we investigate the case where x is of different length than z_i, for any i, and k and k_i are at least 3, for all $1 \leqslant i \leqslant n$, and $n \leqslant k$. We prove that for those equations all solutions are of rank 1, that is, x and z_i are powers of the same word for all $1 \leqslant i \leqslant n$. It is also shown that this result implies a well-known result by Appel and Djorup about the more special case where $k_i = k_j$ for all $1 \leqslant i < j \leqslant n$.

MSC: 68R15; 20M05

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1. Introduction

Word equations of the form

$$x^k = z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n} \tag{1}$$

have long been of interest, see for example [7,5,1]. Originally motivated from questions concerning equations in free groups special cases of (1) in free semigroups were investigated. For example

$$x^k = z_1^{k_1} z_2^{k_2}$$

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is of rank 1 which was shown by Lyndon and Schützenberger [7], and Lentin [5] investigated the solutions of

$$x^k = z_1^{k_1} z_2^{k_2} z_3^{k_3}$$

which has solutions of higher rank, see Example 6, and Appel and Djorup [1] investigated

$$x^k = z_1^k z_2^k \cdots z_n^k .$$

We show in Theorem 5 of this paper that equations of the form (1) are of rank 1, if all exponents are larger than 2 and $n \leqslant k$ and x is not a conjugate of z_i for any $1 \leqslant i \leqslant n$. This result straightforwardly implies Theorem 7 by Appel and Djorup [1].

We continue with fixing some notation. More basic definitions can be found in [6]. Let A be a finite set and A^* be the free monoid generated by A. We call A *alphabet* and the elements of A^* *words*. Let $w = w_{(1)} w_{(2)} \cdots w_{(n)}$ where $w_{(i)}$ is a letter, for every $1 \leqslant i \leqslant n$. We denote the length n of w by $|w|$. An integer $1 \leqslant p \leqslant n$ is a *period* of w, if $w_{(i)} = w_{(i+p)}$ for all $1 \leqslant i \leqslant n - p$. A nonempty word u is called a *border* of a word w, if $w = uv = v'u$ for some suitable words v and v'. We call w *bordered*, if it has a border that is shorter than w, otherwise w is called *unbordered*. A word w is called *primitive* if $w = u^k$ implies that $k = 1$. We call two words u and v *conjugates*, denoted by $u \sim v$, if $u = xy$ and $v = yx$ for some words x and y. Let $[u] = \{v \mid u \sim v\}$ and $w^* = \{w^i \mid i \geqslant 0\}$.

Let Σ be an alphabet. A tuple $(u, v) \in \Sigma^* \times \Sigma^*$ is called *word equation* in Σ, usually denoted by $u = v$. Let $u, v \in \Sigma^*$ be such that every letter of Σ occurs in u or v. A morphism $\varphi \colon \Sigma^* \to A^*$ is called a *solution* of $u = v$, if $\varphi(u) = \varphi(v)$. The *rank of a solution* φ of an equation $u = v$ is the minimum rank of a free subsemigroup that contains $\varphi(\Sigma)$. The *rank of an equation* is the maximum rank of all its solutions.

2. Some known results

The following theorem was shown by Fine and Wilf [2]. As usual, gcd denotes the greatest common devisor.

Theorem 1. *Let $w \in A^*$, and p and q be periods of w. If $|w| \geqslant p + q - \gcd\{p, q\}$ then $\gcd\{p, q\}$ is a period of w.*

The following lemma is a consequence of Theorem 1; see [3].

Lemma 2. *Let $w \in A^*$ and p be the smallest period of w. Then, for any period q of w, with $q \leqslant |w| - p$, we have that q is a multiple of p.*

The following theorem follows Lyndon and Schützenberger's proof [7] for free groups. See also [4] for a short direct proof and the following Lemma 4.

Theorem 3. *Let $x, y, z \in A^*$ and $i, j, k \geqslant 2$. If $x^i = y^j z^k$ then $x, y, z \in w^*$ for some $w \in A^*$.*

Lemma 4. *Let $x, z \in A^*$ be primitive and nonempty words. If z^m is a factor of x^k for some $k, m \geqslant 2$, then either $(m-1)|z| < |x|$ or z and x are conjugates.*

Proof. Assume that $(m-1)|z| \geqslant |x|$. Then z^m has two periods $|x|$ and $|z|$, and hence, a period $\gcd\{|x|, |z|\}$ by Theorem 1. Now, $|x| = |z|$ and x and z are conjugates. $\square$

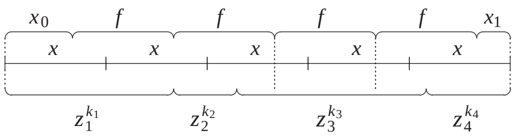
3. The main result

The following theorem is the main result of this paper. It shows that the solutions of a word equation of the form $x^k = z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$ are necessarily of rank 1 under certain conditions.

Theorem 5. *Let $n \geqslant 2$ and $x, z_i \in A^*$ and $|x| \neq |z_i|$ and $k, k_i \geqslant 3$, for all $1 \leqslant i \leqslant n$. If $x^k = z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$ and $n \leqslant k$ then $x, z_i \in w^*$, for some $w \in A^*$ and all $1 \leqslant i \leqslant n$.*

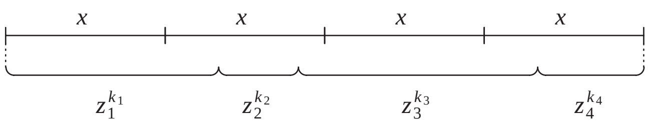
Proof. Assume w.l.o.g. that x, z_i, for all $1 \leqslant i \leqslant n$, are primitive words. Note, that $|z_i^{k_i - 1}| < |x|$ by Lemma 4, and therefore $|z_i| < |x|$ for all i.

If $n < k$ then let f be an unbordered conjugate of x, and $x^k = x_0 f^{k-1} x_1$ with $x = x_0 x_1$. Let us illustrate this case with the following drawing.



By the pigeon hole principle there exists an i such that f is a factor of $z_i^{k_i}$. But now, f is bordered; a contradiction.

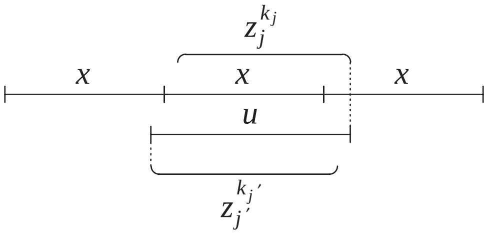
Assume $n = k$ in the following. Let us illustrate this case with the following drawing:



From $k_i \geqslant 3$, for all $1 \leqslant i \leqslant n$, follows that there exists a primitive word $z \in A^*$ such that for every i with $|x| \leqslant |z_i^{k_i}|$ we have that $|z_i|$ is the smallest period of x and $z_i \in [z]$ by Lemma 2.

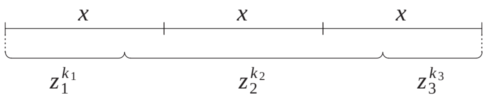
There exists an i such that $|x| \leqslant |z_i^{k_i}|$ by a length argument. We also have for all $1 \leqslant i < n$ that, if $|x| \leqslant |z_i^{k_i}|$ then $|z_{i+1}^{k_{i+1}}| < |x|$, otherwise either z is not primitive or $x \in z_0^*$, with $z_0 \in [z]$, and x is not primitive. Similarly for z_{i-1}. Moreover, we have that all factors $z_j^{k_j}$ with $|x| \leqslant |z_j^{k_j}|$ occur in a word u which is a factor of xxx and $|u| < |x| + |z|$ otherwise z^{k_i+1}, for some $1 \leqslant i \leqslant n$, and xx have a common factor of length greater or equal to $|x| + |z|$

and either x or z is not primitive. Consider the following drawing:



Therefore, we have for every i with $|x| \leqslant |z_i^{k_i}|$ that $|z_{i+1}^{k_{i+1}}| < |zz|$ because $|z_{i+1}| < |z|$ and otherwise z is not primitive. This proves the case for $k > 3$ since then $|z_i^{k_i} z_{i+1}^{k_i}| < |xx|$ (for $k_i \geqslant 3$ for all $1 \leqslant i \leqslant n$ is required), for every i such that $|x| \leqslant |z_i^{k_i}|$, and $|z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}| < |x^k|$; a contradiction.

The case $k = 3$ remains. Since we can construct from one equation a new one of the same rank by cyclic shifts, we can assume that $|x| \leqslant |z_2^{k_2}|$. Let us consider the following drawing for example:



By the arguments above, we have that $|z_1^{k_1}| < |x|$ and $|z_3^{k_3}| < |x|$. Now, $|z^{k_2-1}| < |x| < |z^{k_2}|$ and $|z^{k_2}| < |z_1^{k_1}| + |z_3^{k_3}|$. Let $x = z'^{k_2-1} z'_0$, where $z' \in [z]$ and z'_0 is a prefix of z'. Let g be an unbordered conjugate of z' such that $z'z' = g_1 g g_0$, where $g = g_0 g_1$ and $z' = g_1 g_0$. We get a contradiction, if $|g_1 g| \leqslant |z_1^{k_1}|$ since then $z_1^{k_1}$ covers g, and hence, g is bordered. So, assume $|g_1 g| > |z_1^{k_1}|$. But now, $|z_1^{k_1} z_2^{k_2}| < |xxg_1|$, since $|g_0 z'_0 x| < |z_2^{k_2}| < |x| + |z| < |g_0 z'_0 x g_1|$, and g is covered by $z_3^{k_3}$; a contradiction again. $\square$

The following example shows why the condition $|x| \neq |z_i|$ is needed in Theorem 5.

Example 6. Consider $x^4 = z_1^3 z_2^3 z_3^3$. There exists a solution φ of rank 2 with $\varphi(x) = \varphi(z_1) = a^3 b^3$ and $\varphi(z_2) = a^3$ and $\varphi(z_3) = b^3$.

Theorem 5 implies the following result by Appel and Djorup [1].

Theorem 7. *Let $n \geqslant 2$ and $x, z_i \in A^*$, for all $1 \leqslant i \leqslant n$. If $x^k = z_1^k z_2^k \cdots z_n^k$ with $n \leqslant k$, then $x, z_i \in w^*$, for some $w \in A^*$ and all $1 \leqslant i \leqslant n$.*

Proof. If $n = 2$ the result follows from Theorem 3. Assume $n > 2$ in the following. Let $\bar{x}$ and $\bar{z}_i$ denote the primitive roots of $x = \bar{x}^\ell$ and $z_i = \bar{z}_i^{\ell_i}$, for all $1 \leqslant i \leqslant n$, respectively. Then we have

$$\bar{x}^{\ell k} = \bar{z}_1^{\ell_1 k} \bar{z}_2^{\ell_2 k} \cdots \bar{z}_n^{\ell_n k}\,. \tag{2}$$

If there exists an i such that $|\bar{z}_i| = |\bar{x}|$ then $\bar{z}_i \sim \bar{x}$ and we have the equation

$$\bar{x}^{(\ell-\ell_1)k} = \bar{z}_1^{\ell_1 k} \bar{z}_2^{\ell_2 k} \cdots \bar{z}_{i-1}^{\ell_{i-1} k} \bar{z}_{i+1}^{\ell_{i+1} k} \cdots \bar{z}_n^{\ell_n k}, \tag{3}$$

which has not a higher rank than (2). Since (3) meets our assumptions this reduction can be iterated until either $n = 2$ or $|\bar{z}_i| \neq |\bar{x}|$ for all $1 \leqslant i \leqslant n$. But, then Theorem 5 gives the result. □

References

[1] K.I. Appel, F.M. Djorup, On the equation $z_1^n z_2^n \cdots z_k^n = y^n$ in a free semigroup, Trans. Amer. Math. Soc. 134 (1968) 461–470.
[2] N.J. Fine, H.S. Wilf, Uniqueness theorem for periodic functions, Proc. Amer. Math. Soc. 16 (1965) 109–114.
[3] V. Halava, T. Harju, L. Ilie, Periods and binary words, J. Combin. Theory Ser. A 89 (2) (2000) 298–303.
[4] T. Harju, D. Nowotka, The equation $x^i = y^j z^k$ in a free semigroup, Semigroup Forum 68 (2004) 488–490.
[5] A. Lentin, Sur l'équation $a^M = b^N c^P d^Q$ dans un monoïd libre, C. R. Acad. Sci. Paris 260 (1965) 3242–3244.
[6] M. Lothaire, Combinatorics on Words, volume 12 of Encyclopedia of Mathematics and its Applications, vol. 2, Addison-Wesley, Reading, MA, 1983.
[7] R.C. Lyndon, M.P. Schützenberger, The equation $a^M = b^N c^P$ in a free group, Michigan Math. J. 9 (1962) 289–298.